Generalized Intersection Patterns and Two-Symbol Balanced Arrays

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ABSTRACT

Let m be a fixed integer, $\mathbf{m}=\{0,1,\ldots,m-1\}$; let C be a family of nonvoid subsets of $\mathbf{m}$, and let R be a hereditary subfamily of C. Given finite sets $A_0,\ldots,A_{m-1}$ such that $\bigcap_{i\in B}A_i=\varnothing$ for all $B\subseteq\mathbf{m}$, $B\notin C$, the vector of $|\bigcap_{i\in R_j}A_i|$ $(R_j\in R)$ is called a C-supported R-intersection pattern. The characterization of the Y_{RC} of such patterns is a difficult combinatorial problem even for $m=5$ and simple families R and C. We study the algebraic structure of the convex cone Y_{RC} and its dual, and an integer linear-programming aspect of the problem; in particular we introduce the notion of content and pseudocontent. A relaxation leads to quadratic and higher forms over certain subsets of reals. As an application we study the natural link between highly symmetric patterns and two-symbol balanced arrays.

INTRODUCTION AND PRELIMINARIES

The paper studies general properties of sets of vectors representing the cardinalities of certain intersections of families of finite sets (i.e. edges of hypergraphs). As we wish to include various combinatorial objects like configurations, designs, balanced two-symbol arrays etc., we allow intersections of more than two sets while stipulating that certain intersections are

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required to be empty. The characterization of these vectors, called R-C patterns, and related problems are in general rather hard problems. In this paper we attempt no more than to bring forth the definitions, list some simple properties, establish relations to convex cones and to quadratic and higher forms, study linear-programming relaxations and outline some applications for highly symmetric special cases. The simplest example is provided by intersection patterns and matrices (or equivalently of matrices of cardinalities of symmetric differences of Hamming distances of vertex sets of hypercubes), but these are mentioned only briefly (more information in [7,14,24]). The paper is organized in six sections. The first one introduces the concept of an R-C pattern, lists a few examples, defines the realizability of a given pattern and its integer linear-programming aspect and concludes with some facts needed later. In Sec. 2 the basic problem of realizability is relaxed to the problem of pseudorealizability, where the R-C pseudorealizable vectors are the lattice (integer) points in the convex cone generated by the R-C realizable vectors. We also consider the dual cone of this cone and its facets. In Sec. 3 we introduce and study the content of a pattern (extending the least size of a set on which an intersection pattern may be realized) and the corresponding notion of pseudocontent for pseudorealizable patterns. In Sec. 4 the system of equations whose integer nonnegative solutions are exactly the R-C patterns is transformed to a similar but smaller system of inequalities, and the possibility of generating facets by applying the Fourier-Motzkin elimination method is discussed. In Sec. 5 a gradual restriction of the preceding results leads to the characterization of balanced two-symbol arrays (a generalization of t-designs for statistical purposes) which includes special cases obtained earlier by Srivastava [21]. In the last section we relate the study of pseudorealizable patterns (without emptiness restriction) to the study of quadratic and higher forms assuming nonnegative values of a subset S of $\mathbb{C}$(e.g. $S=\{0,1\}$, $\mathbb{Q}_+$, $\mathbb{R}_+$ etc.). This relaxation, called S-copositivity, includes some well-known problems of positive semidefinite forms, copositive matrices etc.

NOTATION

We indicate by $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ the sets of integers, rationals and reals. Further $\mathbb{N}$, $\mathbb{P}$, $\mathbb{Q}_+$ and $\mathbb{R}_+$ denote the nonnegative integers, positive integers, nonnegative rationals and nonnegative reals. For $m\in\mathbb{P}$ the set $\{0,1,\dots,m-1\}$ is denoted by m. Further, P stands for the family of all nonempty subsets of m. The set P is linearly (totally) ordered by setting $a\leqslant b$ whenever $a,b\in P$, $\sum_{i\in \mathrm{m}}a_i 2^i\leqslant\sum_{i\in\mathrm{m}}b_i 2^i$, where $a_i=1$ if $i\in a$, $a_i=0$ if $i\notin a$, and $b_0,\dots,b_{m-1}$ are defined in a similar way. Note that the set-theoretical inclusion $\subseteq$ is a subrelation of $\leqslant$ (i.e., $a \subseteq b \Rightarrow a \leqslant b$). For $0 < t \leqslant m$ we set

$$P_t = \{a \in P : |a| \leqslant t\}, \qquad P_{(t)} = \{a \in P : |a| = t\}.$$

For our purposes it will be convenient to use column vectors whose coordinates are not indexed by numbers $1, \ldots, n$ but by sets from a family $C \subseteq P$; hence such vectors will be denoted $\langle x_c : c \in C \rangle$ (the order of the coordinates is determined by the ordering $\leqslant$ of the sets c in P). For example the characteristic or incidence vector of a subset A of a set B is the zero-one vector $\chi_B(A) = \langle x_b : b \in B \rangle$ defined by $x_b = 1$ if $b \in A$ and $x_b = 0$ otherwise.

For a set B the symbol $B^{k \times l}$ denotes the set of $k \times l$ matrices over B (e.g. $\mathbb{R}_+^{n \times n}$ denotes all nonnegative real $n \times n$ matrices). The ith row (column) of a matrix A is denoted by A_{i*} (A_{*i}). We write B^k for $B^{k \times 1}$. The row and column vectors of a matrix are often indexed by sets from two subsets R and C of P.

1. R-C PATTERNS

1.1. Let m be a fixed positive integer. Let $\mathcal{A} = \{A_0, \ldots, A_{m-1}\}$ be a family of nonempty (but not necessarily pairwise distinct) finite sets, and let $A = A_0 \cup \cdots \cup A_{m-1} = \{a_0, \ldots, a_{n-1}\}$. To $\mathcal{A}$ associate the binary $m \times n$ zero-one matrix X whose ith row is $\chi_A(A_i)$ ($i \in \mathrm{m}$) (i.e. $X_{ij} = 1$ iff $a_j \in A_i$). The columns of this standard matrix X in their turn may be considered as the characteristic vectors of certain subsets of m. The set B of these subsets is called the *support* of $\mathcal{A}$. If the support of $\mathcal{A}$ is a subset of $C \subseteq P$, we say that $\mathcal{A}$ is *C-supported.* Let R and C be two fixed subsets of P of sizes ρ and γ. A vector $y = \langle y_r : r \in R \rangle$ is a *C-supported R-intersection pattern* (*R-C pattern* for short) if there exists a C-supported family $\{A_0, \ldots, A_{m-1}\}$ such that $y_r = |\bigcap_{i \in r} A_i|$ for each $r \in R$. The R-C patterns will be interchangeably called also the *R-C realizable vectors*, and their set will be denoted Y_{RC} or simply Y. The definition has the following motivation. To each $d \subseteq \mathrm{m}$ assign the element A_d of the Boolean algebra (or field of sets) generated by $\mathcal{A}$ which is the intersection of A_i ($i \in d$) and the complements $A \backslash A_i$ ($i \in \mathrm{m} \backslash d$). Rather obviously the fact that the support of $\mathcal{A}$ is included in C means that A_d is empty for each d not in C. Thus in an R-C pattern we monitor the sizes of certain intersections (determined by R) of a system of sets such that certain elements of the Boolean algebra generated by the system are forcibly empty (these being determined by d not in C). A case of special interest arises when C is *hereditary* (i.e., C is such that $a \in C$ whenever $\varnothing \neq a \subseteq c \in C$). Then in an R-C pattern we monitor the sizes of certain intersections of systems of m finite sets such that certain intersections are void (the former being determined by the sets from R and the latter by the sets not in C). Before proceeding further we mention a few examples of R-C patterns.

EXAMPLE 1.2 [10]. Let $R = P_{(2)}$ and $C = P$. In this case we simply monitor the sizes of intersections of pairs of sets (without any restriction on the support). A moment's reflection shows $Y_{P_{(2)}} = N^{\rho}$ [where $\rho = \frac{1}{2}m(m-1)$], i.e., each integer nonnegative ρ-vector is a $P_{(2)}$-P pattern. The least size of $A_0 \cup \cdots \cup A_{m-1}$ realizing a given $y \in N^{\rho}$ [called later, in Sec. 2, the content $c(y)$ of y] may be of some interest but is not discussed here.

1.3. A slight modification of Example 1.2 produces the following rather difficult problem. Let $R = P_2$ and $C = P$. In this case we monitor the sizes of the sets and their pairwise intersections (again without any restriction on the support). In this case the set Y is the set of so called *intersection patterns* introduced in [5] and studied among others in [7,8] (see [7] for a bibliography of this topic). The set Y has been characterized only for $m \leqslant 4$ (Deza [2,3] and Kelly [14]). For $m = 4$ the set Y can be described by 40 inequalities. Only some inequalities are known for $m > 4$. The intersection patterns can also be viewed as *intersection matrices*. For this we arrange the numbers $|A_i \cap A_j|$ not in a vector but in an $m \times m$ matrix. Using the matrix X from 1.1, we obtain that the set of intersection matrices is

$$\{XX^T : X \text{ zero-one } m \times n \text{ matrix, } n = 1, 2, \ldots\}.$$

(This approach is generalized in Sec. 6.) Another equivalent formulation is based on symmetric differences of sets which, in terms of zero-one vectors, can be formulated as the matrix of all Hamming distances between a set of $m+1$ binary n-vectors.

EXAMPLE 1.4. A *tactical configuration* with parameters m, b, k and r is essentially a binary $m \times b$ matrix whose column and row sums are all k and r, respectively (i.e., the columns are the *characteristic* vectors of blocks, while the rows correspond to points). It can be checked directly that a tactical configuration exists iff $(r, \ldots, r)^T$ is a $P_{(1)}$-$P_{(k)}$ pattern. Thus, in principle, tactical configurations may be regarded as special cases of $P_{(1)}$-$P_{(k)}$ patterns.

EXAMPLE 1.5. A *t-design* $S_{\lambda}(t, k, m)$ can be described as a binary $m \times n$ matrix with all column sums k and such that each $t \times n$ submatrix contains exactly λ columns $(1, \ldots, 1)^T$. Clearly $S_{\lambda}(t, k, m)$ exists iff $(\lambda, \ldots, \lambda)^T$ is an $P_{(t)}$-$P_{(k)}$ pattern.

Now we describe an integer linear-programming aspect of R-C patterns. Let $E = E_{RC} = (e_{rc})$ be the $\rho \times \gamma$ zero-one matrix defined by $e_{rc} = 1$ if $r \subseteq c$ and $e_{rc} = 0$ otherwise ($r \in R$, $c \in C$).

EXAMPLE 1.6. Let $m = 2$, $R = \{\{1\}, \{0,1\}\}$ and $C = P_2 = \{\{0\},\{1\},\{0,1\}\}$. Then

$$E = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Indicate by $E\mathbb{N}^\gamma$ the set $\{Ex : x \in \mathbb{N}^\gamma\}$. We shall prove that $E\mathbb{N}^\gamma$ is precisely the set Y of intersection patterns.

PROPOSITION 1.7. *A vector $y \in \mathbb{N}^\gamma$ is an R-C pattern if and only if $y = E_{RC}x$ for some $x \in \mathbb{N}^\gamma$.*

Proof. Necessity: Let X be a zero-one matrix corresponding to y (as introduced in 1.1). Let $r \in R$, and let the jth column X_{*j} of X satisfy $X_{*j} = \chi(c)$ (where $c \in C$, since X is C-supported). Then there is a nonzero contribution to y_r from X_{*j} iff $c \supseteq r$. Denoting by x_c the frequency of $\chi(c)$ in X, we get $y_r = \sum_{r \subseteq c \in C} x_c$, or equivalently, $y_r = E_{r*}x$, proving $y = Ex$.

Sufficiency: Let $x = \langle x_c : c \in C \rangle \in \mathbb{N}^\gamma$, and let X be a zero-one matrix having exactly x_c columns $\chi(c)$. The argument above can be reversed to show that $y = Ex$ corresponds to X, proving that y is an R-C pattern. ∎

EXAMPLE 1.8. In the situation of Example 1.6 the set of patterns is $Y = E\mathbb{N}^2 = \{(x_2 + x_3, x_3)^T : x_2, x_3 \in \mathbb{N}\}$, which indeed monitors the sizes of A_1 and $A_0 \cap A_1$ (without any restriction on the support).

1.9. A few comments on possible redundancies in the matrix E. Firstly we may remove the elements of R and C corresponding to zero rows and columns of E, because the former lead to coordinate places which are zero in all $y \in Y$, while the latter have no impact on Y whatsoever.

It may happen that E has two identical columns. Without any loss of generality the two corresponding variables x_i and x_j may be merged into a single one (equal to the sum $x_i + x_j$).

Identical rows lead to a duplication of coordinates in intersection patterns only and so may also be removed. These reductions lead to the following pairs R-C. For $p \in P$ set $(p] = \{q \in P : q \subseteq p\}$ and $[p) = \{q \in P : p \subseteq q\}$. The pair R-C is said to be *reduced* if both the families $\{(c] \cap R : c \in C\}$ and $\{[r) \cap C : r \in R\}$ consist of pairwise distinct nonempty sets. A simple check shows that E has pairwise distinct and nonzero rows and pairwise distinct and nonzero columns precisely if R-C is reduced. From now we assume that R-C is reduced (unless indicated to the contrary).

We note some immediate consequences of Proposition 1.7 (most of the discussion below holds even if we replace the binary matrix E by any integer nonnegative matrix). The set $\mathbb{N}^{\rho}$ with the componentwise (i.e. vector) addition forms clearly an Abelian (cancellative) monoid (i.e., it contains $(0,\ldots,0)^T$ and is closed under addition).

REMARK 1.10. The set Y of R-C patterns is a submonoid of $\mathbb{N}^{\rho}$ generated by the column vectors $E_{*0},\ldots,E_{*l-1}$ of E. In particular, Y contains all nonnegative integer multiples of its elements.

It is natural to ask whether these column vectors form a minimal set of generators for Y (i.e. whether there is no redundancy):

PROPOSITION 1.11. *The column E_{*c} belongs to the monoid generated by the other columns of E if and only if there is $B \subseteq C\setminus\{c\}$ such that for each $r \in R$*

$$|[r) \cap B| = |[r) \cap \{c\}|.$$

Proof. Necessity: Since E is binary, E_{*c} is not only an integer nonnegative linear combination of other columns but even their sum, i.e., $E_{*c} = \Sigma_{j\in A} E_{*j}$ for some $A \subseteq C\setminus\{c\}$. Let $r \in R$. Clearly $e_{rc}=0$ implies $e_{rj}=0$ for all $j \in A$. Moreover, if $e_{rc}=1$, then $e_{rj}=1$ for a unique $j \in A$. Since $e_{rc}=|[r)\cap \{c\}|$ by the definition of E, it suffices to set $B=A$.

Sufficiency: Reversing the above argument, we obtain that E_{*c} is the sum of E_{*j}'s with $j \in B$. ∎

REMARK 1.12. We say that R and C form a *correct pair* if E_{RC} is fully dimensional [i.e., $\operatorname{rank} E_{RC} = \min(\rho, \gamma)$].

For a given family $\mathcal{A} = \{A_0,\ldots,A_{m-1}\}$ of sets and $p \in P$ it is sometimes possible to express the size of $\bigcap_{i\in p} A_i$ in terms of the corresponding R-C pattern y even if $p \notin R$. Set $\chi_{pc}=1$ if $p \subseteq c$ and $\chi_{pc}=0$ otherwise.

PROPOSITION 1.13. *Let $p \in P$, let $\mathcal{A} = \{A_0,\ldots,A_{m-1}\}$ be a C-supported family of sets, and let y be the corresponding R-C pattern. If a_r $(r \in R)$*

satisfy the equations

$$\sum_{c \supseteq r \in R} a_r = \chi_{pc} \qquad (c \in C), \tag{1.1}$$

then

$$\left| \bigcap_{i \in p} A_i \right| = \sum_{r \in R} a_r y_r.$$

Proof. Set $a = \langle a_r : r \in R \rangle^T$, and let $y = Ex$ where x corresponds to $\mathscr{C}$. Then

$$ay = (aE)x = \sum_{c \in C} \left(\sum_{c \supseteq r \in R} a_r \right) x_c = \sum_{c \in C} \chi_{pc} x_c$$

$$= \sum_{p \subseteq c \in C} x_c = \left| \bigcap_{i \in p} A_i \right|. \qquad \blacksquare$$

EXAMPLE 1.14. Let $1 < t < k < m$, and let X be a matrix corresponding to a $P_{(t)}$-$P_{(k)}$ pattern y. For $p = \{i\}$ and $r \in P_{(t)}$ set

$$a_r = \binom{k-1}{t-1}^{-1} \quad (i \in r); \qquad a_r = 0 \quad (i \notin r),$$

Then (1.1) hold, and therefore the row sum of the ith row of X $(=|A_i|)$ equals $\binom{k-1}{t-1}^{-1} \sigma_i$, where

$$\sigma_i = \sum_{i \in r \in P_{(t)}} y_r.$$

In particular, we see that each σ_i is divisible by $\binom{k-1}{t-1}$, and if $\sigma_0 = \cdots = \sigma_{m-1}$, then X has constant row sums. Thus, if X is a t-design $S_\lambda(t,k,m)$, then $y_r = \lambda$ for all $r \in P_{(t)}$ implies the well-known fact: X has constant row sums

$$\binom{m-1}{t-1} \binom{k-1}{t-1}^{-1} \lambda.$$

Note that X may have constant row sums even if the y_r's are not all identical [e.g. if $t=2$, $m=4$ and $y_{\{0,1\}}=y_{\{2,3\}}=(k-1)a$, $y_{\{0,2\}}=y_{\{1,3\}}=(k-1)b$, $y_{\{0,3\}}=y_{\{1,2\}}=(k-1)c$, where $a,b.c\in\mathbb{N}$].

2. PSEUDOREALIZABLE VECTORS AND DUAL CONES

It is natural to ask about the affine dimension of Y in the Euclidean space $\mathbb{R}^\rho$ [which, due to $(0,\ldots,0)^T\in Y$, equals the maximum cardinality of a linearly independent subset of Y]. Using the γ-vectors $(1,0,\ldots,0)^T$, $\ldots,(0,\ldots,0,1)^T$ it is easy to verify that the affine dimension of Y equals $\operatorname{rank}E$. These and other considerations lead naturally to the *convex hull*, $\operatorname{conv}Y$, of Y (the least convex set containing Y). Clearly $\operatorname{conv}Y=E\mathbb{R}^\gamma_+$, and therefore $\operatorname{conv}Y$ is a pointed polyhedral (or regular) cone with apex $(0,\ldots,0)^T$. Its rational points are characterized as follows:

PROPOSITION 2.1. *We have*

$$\mathbb{Q}^\rho_+ \cap \operatorname{conv}Y = E\mathbb{Q}^\gamma_+ = \bigcup_{u=1}^{\infty} u^{-1}Y.$$

Proof. Let $z\in\mathbb{Q}^\rho_+\cap\operatorname{conv}Y$. Then we can find a positive integer p, vectors $y_i\in Y$ and $\lambda_i\in[0,1]$ $(i\in\mathbf{p})$ such that

$$\sum_{i\in p}\lambda_i y_i = z, \qquad \sum_{i\in p}\lambda_i = 1.$$

This system of linear equations for $\lambda_0,\ldots,\lambda_{p-1}$ with rational coefficients, being solvable, is also solvable by some rationals $\lambda_i'=a_ib_i^{-1}$ $(i\in\mathbf{p})$, a_i nonnegative and b_i positive integers. Thus for $u=b_0\cdots b_{p-1}$ the element uz is a nonnegative integer combination of the vectors $y_0,\ldots,y_{p-1}\in Y$ and as such belongs to Y. This proves $\mathbb{Q}^\rho_+\cap\operatorname{conv}Y\subseteq\bigcup_{u=1}^\infty u^{-1}Y$. The inclusion of the latter in $E\mathbb{Q}^\rho_+$ being obvious, it suffices to show that $E\mathbb{Q}^\rho_+$ is included in $\operatorname{conv}Y$. However, this follows as above from the fact that a rational nonnegative combination of the columns of E is between the origin and an element of Y. ∎

An *extreme ray* of $\operatorname{conv}Y$ is a halfline H in $\operatorname{conv}Y$ sharing points with no line segment between two points from $\operatorname{conv}Y\setminus H$. Clearly each extreme ray is of the form $E_{*c}\mathbb{R}_+$ (i.e., the halfline through a column vector of E), but it may happen that a vector E_{*c} does not determine an extreme ray (i.e., E_{*c} is a nonnegative linear combination of other column vectors of E). In order to give a sufficient condition for such redundancy we need the following definition. Let $H \subseteq P$, and let $h', h'' \in H$. We say that h' and h'' are *up-and-down linked* if there exists n and $h_1, \ldots, h_{2n-1} \in H$ such that $h' \supseteq h_1 \subseteq h_2 \supseteq h_3 \subseteq \cdots \supseteq h_{2n-1} \subseteq h''$ (i.e., there is an n-sequence in H such that the intersection of consecutive elements contains an element from H). An *up-and-down component* of H is a maximal subset M of H whose elements are pairwise up-and-down linked (here maximal means not included in any other such set, i.e., $h' \in M$ is up-and-down linked to no $h'' \notin M$). Recall that $(c] \cap R$ means the set of all $r \in R$ included in c. A sufficient condition for extreme rays is based on the following lemma.

LEMMA 2.2. *If the column vector E_{*c} of E is not on an extreme ray of* $\operatorname{conv} Y$, *then there is $A \subseteq C \setminus \{c\}$ such that*

(i) *$r \subseteq a$ implies $r \subseteq c$ whenever $a \in A$ and $r \in R$,*

(ii) *The union of each up-and-down component of $(c] \cap R$ is contained in at least one element of A. If, moreover, the columns of E are incomparable (i.e. for $\{c', c''\} \subseteq C$ there are $r', r'' \in R$ such that $r' \subseteq c', r' \nsubseteq c'', r'' \subseteq c'$ and $r'' \nsubseteq c''$), then $(c] \cap R$ is contained in at least two elements of A.*

Proof. Let

$$E_{*c} = \sum_{a \in A} E_{*a} x_a, \tag{2.1}$$

where $A \subseteq C \setminus \{c\}$ and all x_a are positive reals. For each $r \in R$, clearly $e_{rc} = 0$ implies $e_{ra} = 0$ for all $a \in A$, which, by the definition of E, is condition (i). For $r \in R$ contained in c, set $T_r = \{a \in A : r \subseteq a\}$ and obtain from (2.1)

$$\sum_{a \in T_r} x_a = e_{rc} = 1. \tag{2.2}$$

Consider $r', r'' \in \mathrm{R}$, $r' \subseteq r''$. Then $T_{r'} \supseteq T_{r''}$ and from (2.2)

$$0 = \sum_{a \in T_r} x_a - \sum_{a \in T_r} x_a = \sum_{a \in T_r \setminus T_r} x_a.$$

In view of $x_a > 0$ we have $T_{r'} \setminus T_{r''} = \varnothing$ and hence $T_{r'} = T_{r''}$. Thus $T_r = T_{r''}$

whenever r' and r'' are up-and-down linked in $(c] \cap R$. To prove (ii) we must show that T_r is not a singleton whenever the columns of E are incomparable. Indeed, were $T_r = \{h\}$, then (2.2) implies $x_h = 1$ and from (2.1) we get the contradiction $E_{*c} \geqslant E_{*h}$. ■

Now we can prove a sufficient condition (satisfied in several examples) for extreme rays. We say that $H \subseteq R$ is *connected* if each pair of its elements is linked by a sequence of elements of H having nonempty intersections of consecutive elements. A *connected component* of H is a maximal connected subset of H. We have:

PROPOSITION 2.3. *Let R contain all singletons from* m, *and let $c \in C$. If the union u of each connected component of $(c] \cap R$ satisfies $u \subseteq c' \subset c$ for at most one $c' \in C$, then E_{*c} determines an extreme ray of* conv Y.

Proof. Suppose E_{*c} does not determine an extreme ray. Let A be the set from Lemma 2.2. Since R contains all singletons, condition (i) from Lemma 2.2 yields $a \subseteq c$ for all $a \in A$. For the same reason up-and-down connectedness agrees with connectedness and the columns of E are incomparable. Thus we obtain a contradiction to (ii) for Lemma 2.2. ■

We are primarily interested in integer vectors from conv Y which are called the R-C *pseudorealizable* vectors. The set of R-C pseudorealizable vectors may not agree with Y. Indeed, it may happen that there is $x \in \mathbb{N}^Y$ and $\lambda \in \mathbb{Q}_+$ such that λEx is an integer vector while λx is not. For intersection patterns this happens already for $m = 5$. The following example was given in [2] (for distance matrices) and later in [14]. The sets $A_0 = \{0,1,\ldots,7\}$, $A_1 = \{0,1,2,3\}$, $A_2 = \{0,1,4,5\}$, $A_3 = \{1,2,4,6\}$ and $A_4 = \{2,3,4,5\}$ yield the intersection pattern $z = \langle 8,4,4,4,4,2,4,4,2,2,4,4,2,2,2\rangle$ (listed in the order $|A_0|, |A_1|, |A_0 \cap A_1|, |A_2|, |A_0 \cap A_2|, \ldots, |A_3 \cap A_4|$). It can be verified that the integer vector $\frac{1}{2}z$ is not an intersection pattern. This example shows that, in general, Y cannot be described as the set of integer points of a polyhedral cone determined by a system of linear inequalities. In this respect our problem differs from several other combinatorial problems. However even the knowledge of conv Y would provide a good start for the characterization of Y. Since conv Y is a pointed polyhedral cone with apex 0 there is a unique set $D = D_{RC}$ of real row ρ-vectors such that for every $d \in D$ we have $dy \geqslant 0$ whenever $y \in Y$. The set D is the *dual* or *polar cone* to conv Y and corresponds to the set of all homogeneous linear inequalities $dy \geqslant 0$ valid for all the vectors y from Y. Thus an economical description of D provides a good description of conv Y. We have:

PROPOSITION 2.4. *Let d be a real row ρ-vector. Then $d \in D$ if and only if dE is nonnegative.*

Proof. By definition, $d \in D$ is equivalent to $dy \geqslant 0$ for all $y \in Y = E\mathbb{N}^\gamma$. Choosing $y = E(0,\ldots,0,1,0,\ldots,0)^T$ we obtain $dE_{*t} \geqslant 0$ for all columns E_{*t} of E. Conversely, if dE is nonnegative, so is dEx for every $x \in \mathbb{N}^\gamma$. ∎

To $d \in D$ we can associate the hyperplane $H = \{x \in \mathbb{R}^\rho : dx = 0\}$ of the Euclidean space $\mathbb{R}^\rho$. The set $H \cap Y = \{y \in Y : dy = 0\}$ is denoted by F_d. From Proposition 2.4, we get:

Proposition 2.5. *For $d \in D$*

$$F_d = \left\{Ex : x \in \mathbb{N}^\gamma,\ x_c = 0 \text{ whenever } dE_{*c} > 0\right\}.$$

For $d \in D$ let $C' = \{c \in C : dE_{*c} = 0\}$, $\delta = |C'|$ and let $E^{(d)}$ be the $\rho \times \delta$ submatrix of E obtained by deleting all columns E_{*c} such that $dE_{*c} > 0$. Then $F_d = E^{(d)}\mathbb{N}^\delta$. Thus the set F_d is the set $Y_{RC'}$ of all R-C' patterns for some family C' of subsets of $\mathbf{m}$ (however, the pair R-C' may not be reduced). We are interested in the tersest possible description of $\operatorname{conv} Y$. For this it suffices to consider only facets of $\operatorname{conv} Y$. By definition these are all $d \in D$ such that F_d has dimension $\dim Y - 1$. We have:

Proposition 2.6. *Let $d \in D$. Then d is a facet of* $\operatorname{conv} Y$ *if and only if* $\operatorname{rank} E^{(d)} = \operatorname{rank} E - 1$.

Proof. It was mentioned at the beginning of this section that $\dim Y = \operatorname{rank} E$. By the same token, from Proposition 2.5 we obtain $\dim F_d = \dim Y_{RC'} = \operatorname{rank} E^{(d)}$. ∎

2.7. We conclude with an example of problems whose pseudorealizable vectors are easily described while the determination of realizable vectors is difficult. With our notation an (m,t,r,λ)-PBIBD is essentially an $m \times n$ zero-one matrix whose row and column sums are all r and t, respectively, and such that all the scalar products of distinct rows equal λ (i.e., two rows share exactly λ ones in the same positions). Set

$$y = \langle y_r : r \in P_2 \rangle; \qquad y_{\{i\}} = r \quad (i \in \mathbf{m}); \qquad y_{\{i,j\}} = \lambda \quad (0 \leqslant i < j < m). \tag{2.3}$$

From Example 1.5 we obtain: *The following three statements are equivalent:* (i) X *is an* (m,t,r,λ)-PBIBD, (ii) X *is an* $S_\lambda(2,t,m)$, *and* (iii) y *is a* P_2-$P_{(t)}$ *pattern.*

The parameters (m,k,r,λ) and n are related by the well-known equations (see e.g. [18, Chapter 8, Sec. 1]): $t(t-1)n = m(m-1)\lambda$ and $mr = nt$. The elimination of n yields $(t-1)r=(m-1)\lambda$. While this equation does not characterize y as a P_2-$P_{(t)}$ realizable vector, it singles it out as a P_2-$P_{(t)}$ pseudorealizable vector.

PROPOSITION 2.8. *The vector y from* (2.3) *is a P_2-$P_{(t)}$ pseudorealizable vector if and only if $(t-1)r=(m-1)\lambda$.*

Proof. Necessity: For each y there exists $q \in \mathbb{P}$ such that qy is a P_2-$P_{(t)}$ pattern. By what has been shown above we have $(t-1)qr=(m-1)\lambda r$ and therefore $(t-1)q=(m-1)\lambda$.

Sufficiency: Set

$$n = \binom{m}{t}\lambda$$

and let X be the $m \times n$ zero-one matrix in which each column with t ones and $m-t$ zeros appears exactly λ times. Clearly the row sums of X are

$$r' = \lambda\binom{m-1}{t-1},$$

the scalar product of two distinct rows is

$$\lambda' = \lambda\binom{m-2}{t-2},$$

and therefore X is an (m,t,r',λ')-PBIBD. Using the premiss we get

$$r' = \lambda\binom{m-2}{t-2}\frac{m-1}{t-1} = \binom{m-2}{t-2}r;$$

hence the vector y' corresponding to X equals $\binom{m-2}{t-2}y$, proving y to be P_2-$P_{(t)}$ pseudorealizable. ∎

3. CONTENT AND PSEUDOCONTENT

3.1. Let y be an R-C pattern. By Proposition 1.7 there exists $x \in \mathbb{N}^\gamma$ such that $y=Ex$. The vector x need not to be unique, and so it may be interesting to know what is the least value of $x_0+\cdots+x_{\gamma-1}$ for all $x \in \mathbb{N}^\gamma$ satisfying $y=Ex$. This yields a typical integer linear-programming problem:

minimize $\sum_{c\in C} x_c$ such that $Ex = y$, $x \in \mathbb{N}^{\gamma}$. Clearly this program is feasible (i.e., $Ex = y$ is solvable in $\mathbb{N}^{\gamma}$) iff $y \in Y$. For $y \in Y$ the least value $c(y)$ of this program is called the *content* of y (for intersection patterns this concept was introduced in [3, 4, 14]). In other words, for $y \in Y$ the content $c(y)$ is the minimum size of $A_0 \cup \cdots \cup A_{m-1}$ over the families $\{A_0, \ldots, A_{m-1}\}$ having y as their R-C pattern. It follows from Remark 1.10 that c is a subadditive function from Y into $\mathbb{N}$ [i.e., $c(y_1 + y_2) \leqslant c(y_1) + c(y_2)$]; for intersection patterns this is noted in [14]. Some information on realizability of y and the content $c(y)$ may be obtained by relaxation. In this standard integer programming approach we replace the restriction $x \in \mathbb{N}^{\gamma}$ either by $x \in \mathbb{R}^{\gamma}_{+}$ or by $x \in \mathbb{Z}^{\gamma}$. In the first case we get a linear-programming problem (for intersection patterns this idea was brought forward in [14]). Its dual problem is maximize $z \cdot y$ subject to $z \cdot E \leqslant (1, \ldots, 1)$, z a real row ρ-vector. If both the relaxed and dual problems have solutions (are feasible) and are bounded, then the optimal value $z^0 \cdot y$ of the dual does not exceed $c(y)$. (For the special case of t-designs the integer relaxation is studied in [12].)

We have also the following simple bounds. Let d be the largest of the column sums of E. For $y \in Y$ multiply both sides of $y = Ex$ from the left by the row ρ-vector $i = (1, \ldots, 1)$, replace iE by its lower bound i and upper bound di and obtain

$$d^{-1} \sum_{r\in R} y_r \leqslant c(y) \leqslant \sum_{r\in R} y_r.$$

We derive a condition for a lower bound of $c(y)$. Recall that $\chi_{\mathrm{m}}(c)$ is the characteristic vector of c in m.

PROPOSITION 3.3. *Let $R_i \subseteq R$ $(i \in \mathrm{m})$, and suppose that for every $r \in R_i$ there exists an integer m-vector v_{ri} such that for each $c \in C$ the vector $\sum_{c\supseteq r\in R_i} v_{ri}$ is a multiple of the vector $\chi_{\mathrm{m}}(c)$. Let y be an R-C pattern, and let A be the $m\times m$ matrix whose ith column is $\sum_{r\in R_i} y_r v_{ri}$ $(i \in \mathrm{m})$. If A is nonsingular, then $c(y) \geqslant m$.*

Proof. Let $y = Ex$, where $x \in \mathbb{N}^{\gamma}$ satisfies $c(y) = \sum_{c\in C} x_c$. Denote by X the corresponding zero-one $m \times c(y)$ matrix. To prove $c(y) \geqslant m$ it suffices to show that the rows of X are linearly independent. Suppose a real line m-vector z satisfies $zX = 0$. Then $\sum_{i\in c} z_i = 0$ or, equivalently, $z\chi_{\mathrm{m}}(c) = 0$ for all $c \in C$. By definition the ith column of A equals

$$\sum_{r\in R_i} v_{ri} y_r = \sum_{r\in R_i} v_{ri} \sum_{r\subseteq c\in C} x_c = \sum_{c\in C} x_c \sum_{c\supseteq r\in R_i} v_{ri},$$

hence by assumption the ith column of A is a linear combination of the vectors $\chi_m(c)$ and consequently $zA=0$. ■

EXAMPLE 3.4. Consider the intersection patterns (i.e. $R=P_2$ and $C=P$). We may choose $R_i=\{\{i,j\}:j\in m\}$ (where $\{i,i\}$ stands for $\{i\}$) and $v_{\{i,j\}i}=(0,\dots,0,1,0,\dots,0)$ (the jth unit m-vector). Then $A=(y_{\{i,j\}})$.

Divisibility may be used to establish the nonsingularity of A. For example, we have the following result.

Let $A_0,\dots,A_{m-1}$ be finite sets. If there exists a prime p dividing all $|A_i\cap A_j|(0\leqslant i<j<m)$ while dividing no $|A_i|(0\leqslant i<m)$, then $|A_0\cup\cdots\cup A_{m-1}|\geqslant m$. Indeed, $A=(y_{\{i,j\}})$ reduced mod p is a diagonal matrix with nonzero diagonal; hence $\det A=y_{\{0\}}\cdots y_{\{m-1\}}\not\equiv 0 \pmod p$, proving $\det A\neq 0$. The special case $|A_i\cap A_j|\in\{0,l\}$, $|A_i|=k$, l does not divide k, was considered in [9]. More general results of this kind are in [23].

The content is related to the following pattern extension obtained by adding one more set. Let R^* be a subset of R such that each $c\in C$ contains at least one $r^*\in R^*$. Let

$$R'=R\cup\{\{m\}\cup r^*:r^*\in R^*\cup\{\varnothing\}\},\ C'=\{c\cup\{m\}:c\in C\cup\{\varnothing\}\}.$$

Given an R-C pattern y and a nonnegative integer q, define the extension $y[q]$ of y as the vector $z=\langle z_{r'}:r'\in R'\rangle$ satisfying $z_r=y_r$ $(r\in R)$, $z_{r^*\cup\{m\}}=y_{r^*}$ $(r^*\in R^*)$ and $z_{\{m\}}=q$. The following proposition gives a condition for $y[q]$ to be an R'-C' pattern.

PROPOSITION 3.5. *The following statements are equivalent for an R-C pattern y and $q\in\mathbb{N}$:*

(i) *$y[q]$ is an R'-C' pattern,*
(ii) *$y[q]$ is an R'-$(C\cup C')$ pattern,*
(iii) *$q\geqslant c(y)$.*

Proof. (ii)⇒(i): Let $z=y[q]=E_{R',C\cup C'}x$ for a nonnegative integer vector x. For $r^*\in R^*$ by definition

$$\sum_{r^*\subseteq c} x_c=z_{r^*}=z_{r^*\cup\{m\}}=\sum_{r^*\cup\{m\}\subseteq c} x_c$$

(both summations over $C\cup C'$); hence canceling and using the nonnegativity of x, we obtain $x_c=0$ whenever $\{m\}\not\subseteq c\supseteq r^*$, i.e. whenever $r^*\subseteq c\in C$. By definition each $c\in C$ contains an r^*, proving $x_c=0$ for all $c\in C$ and hence (i).

(i)⇒(iii): Let $z=y[q]=E_{R'C'}x'$. Since $\{m\}\subseteq c'$ for all $c'\in C'$, we have $z_{\{m\}}=\sum_{c'\in C'}x'_{c'}$. Thus there exists a zero-one $(m+1)\times q$ matrix X corresponding to $y[q]$. If we drop its last row we obtain a matrix of a C-supported family whose R-C pattern is exactly y, proving $q\geqslant c(y)$.

(iii)⇒(ii): Let $y=E_{RC}x$, where $\sum_{r\in R}x_r=c(y)$. Set $x'_c=0$, $x'_{c\cup\{m\}}=x_c$ and $x'_{\{m\}}=q-c(y)$. A direct check shows that the nonnegative integer vector $x'=\langle x_d : d\in C\cup C'\rangle$ satisfies $y[q]=E_{R',C\cup C'}x'$, proving (ii). ∎

Although for $q<c(y)$ the vector $y[q]$ is not R'-C' realizable, it may still be R'-C' pseudorealizable. Thus define the *pseudocontent* $w(y)$ as the least n such that $y[n]$ is an R'-C' pseudorealizable vector. In other words $w(y)$ is the least $n\in\mathbb{N}$ such that $u_n y[n]$ is an R'-C' pattern for some positive integer u_n (Proposition 2.1). Although $y[w(y)]$ is R'-C' pseudorealizable (and $y[c(y)]$ is R'-C' realizable), it is not evident that $y[q]$ is pseudorealizable for $w(y)\leqslant q\leqslant c(y)$.

PROPOSITION 3.6. *The following statements are equivalent for an R-C pattern y and $q\in\mathbb{N}$:*

(i) *$y[q]$ is R'-C' pseudorealizable,*
(ii) *$y[q]$ is R'-$C\cup C'$ pseudorealizable,*
(iii) *$q\geqslant w(y)$.*

Proof. (ii)⇒(i): Proposition 3.5. (i)⇒(iii): Definition. (iii)⇒(i): Set $w=w(y)$. By definition there is a positive integer u such that $u\cdot y[w]$ is an R'-C' pattern. From $u\cdot y[w]=(uy)[uw]$ (where the latter belongs to the pattern uy and integer uw) and Proposition 3.5 it follows that $u\cdot y[q]=(uy)[uq]$ is an R'-C', pattern proving that $y[q]$ is R'-C' pseudorealizable. ∎

We list some properties of w. As usual $\lceil x\rceil$ denotes the least integer not less than x.

PROPOSITION 3.7. *The function w is a subadditive function satisfying*

$$w(y)=\min\{\lceil c(uy)/u\rceil : u=1,2,\dots\}. \tag{3.1}$$

Proof. Let y_i be R-C patterns $(i=1,2)$. Set $w_i=w(y_i)$. There exist positive integers u_i such that $u_i\cdot y_i[w_i]$ are R'-C' patterns. By a direct check

$$z=(y_1+y_2)[w_1+w_2]=y_1[w_1]+y_2[w_2];$$

hence $u_1u_2z=u_2u_1\,y_1[w_1]+u_1u_2\,y_2[w_2]$, being the sum of two R'-C' realizable vectors, is an R'-C' pattern. Thus z is R'-C' pseudorealizable, whence

$w(y_1+y_2) \leqslant w_1+w_2$, proving the subadditivity of w. To prove the formula, let $w=w(y)$, and let p denote the right side of (3.1). Let $u \in \mathbb{P}$ be such that $u \cdot y[w]$ is R'-C' realizable. From $u \cdot y[w]=(uy)[uw]$ and Proposition 3.5 we get $uw \geqslant c(uy)$ and therefore $w \geqslant \lceil c(uy)/u \rceil \geqslant p$. For the reverse inequality let u_0 be an integer for which $p=\lceil c(u_0 y)/u_0 \rceil$. Then $u_0 p \geqslant c(u_0 y)$; hence $(u_0 y)[u_0 p]$ is R'-C' realizable. In view of $u_0 \cdot y[p]=(u_0 y)[u_0 p]$, we obtain that $y[p]$ is R'-C' pseudorealizable, proving $p \geqslant w$. ∎

4. TRANSFORMATION TO INEQUALITIES AND THE FOURIER-MOTZKIN METHOD

Suppose that $\rho \leqslant \gamma$ and that E has rank ρ. Let $y \in Y$. In the system of linear equations $Ex=y$ we may eliminate certain ρ variables x_b $(b \in B)$ (where $|B|=\rho$). Setting $x_B=\langle x_b : b \in B\rangle^T$ and

$$x_N=\langle x_c : c \in C \setminus B\rangle^T=\langle x_{n_0},\dots,x_{n_{\gamma-\rho-1}}\rangle^T,$$

this elimination results in

$$Fx_N - Gy = x_B \tag{4.1}$$

for suitable $\rho \times (\gamma-\rho)$ and $\rho \times \rho$ rational matrices F and G. If F and G happen to be both integer we say that (4.1) is a *separation* of $y=Ex$ and that R-C is a *separable pair*. The separation may be viewed as a Möbius inversion (or a kind of discrete Fourier transform). Separable matrices are not uncommon. We have:

PROPOSITION 4.1. *If $R \subseteq C$, then E is separable.*

Proof. By definition $e_{rc}=1$ implies $r \subseteq c$ and, as $\subseteq$ is a subrelation of the order $\leqslant$ on m, also $r \leqslant c$. Thus e_{rr} is the first nonzero element of the rth row. The $\rho \times \rho$ submatrix H of E consisting of the columns indexed by elements from R is thus upper diagonal with entries 1 on the diagonal. A simple application of Cramer's rule completes the proof. ∎

There may be several other eliminations with integer matrices; in fact in Sec. 5 we use an elimination different from the above. If F and G are integer, then x_B satisfying (4.1) is integer whenever y and x_N are. However, the nonnegativity of x_B is not granted. Clearly we obtain:

PROPOSITION 4.2. *Suppose E is separable and $Ex=y$ is transformed into (4.1) with integer F and G. Then y is an R-C realizable vector if and only if* *there is* $x_N \in \mathbb{N}^{\gamma-\rho}$ *such that*

$$Fx_N \geqslant Gy. \tag{4.2}$$

In other words, for a separable E, the fact that $y \in Y$ amounts to the solvability (feasibility) of the system of inequalities (4.2) by nonnegative integers.

4.3 For a separable E in principle the dual cone D can be obtained by Fourier-Motzkin elimination. The idea is to eliminate successively the variables $x_{n_0}, \ldots, x_{n_{\gamma-\rho-1}}$ in $Fx_N - Gy \geqslant 0$ until we obtain an equivalent system of inequalities involving y only. The basic idea of the elimination is the following. Suppose we have a system $Az \geqslant 0$ of k inequalities in variables $z = (z_0, \ldots, z_{q-1})^T$ (where 0 is the zero column k-vector). Assume that we have rearranged the inequalities so that the first column $a^T = A_{*1}$ has the form $(a_0, \ldots, a_{i-1}, -a_i, \ldots, -a_{j-1}, 0, \ldots, 0)$, where $0 \leqslant i \leqslant j$ and $a_0, \ldots, a_{j-1}$ are all positive. Let A' be the matrix obtained from A by deleting the first column. For a real vector $z = (z_0, \ldots, z_{q-1})^T$ set $z^{(1)} = (z_1, \ldots, z_{q-1})^T$. The elimination is very simple if either $i = 0$ or $i = j$ (i.e. if the entries in the first column of A are either all nonpositive or all nonnegative). Thus suppose that $0 < i < j \leqslant k - 1$. For $0 \leqslant u < i < v < j$ let b_{uv} be the least common multiple of a_u and a_v, let $f_{uv} = a_u^{-1} b_{uv}$, and let $g_{uv} = a_v^{-1} b_{uv}$. Multiplying the uth (vth) inequality by f_{uv} (g_{uv}), we obtain the integer inequalities

$$-f_{uv} A'_{u*} z^{(1)} \leqslant b_{uv} z_0 \leqslant g_{uv} A'_{v*} z^{(1)}. \tag{4.3}$$

For the time being we are not interested in integrality; hence dividing by b_{uv} we get

$$-a_u^{-1} A'_{u*} z^{(1)} \leqslant z_0 \leqslant a_v^{-1} A'_{v*} z^{(1)}.$$

Thus $-a_u^{-1} A_{u*} z^{(1)} \leqslant a_v^{-1} A'_v z^{(1)}$, i.e.

$$(a_u A'_{v*} + a_v A'_{u*}) z^{(1)} \geqslant 0. \tag{4.4}$$

These inequalities combined with the inequalities $A'_{w*} z' \geqslant 0$ $(w = j, \ldots, q-1)$ (i.e. original inequalities not containing z_0) form a system of inequalities which is denoted $A^{(1)} z^{(1)} \geqslant 0$. It is not difficult to see that $Az \geqslant 0$ is solvable iff

$A^{(1)}z^{(1)} \geqslant 0$ is solvable. If this is the case, the first coordinate z_0 satisfies

$$\max\left\{-a_u^{-1}A'_{u\bullet}z^{(1)} : 0 \leqslant u < i\right\} \leqslant z_0 \leqslant \min\left\{a_v^{-1}A'_{v\bullet}z^{(1)} : i \leqslant v < j\right\}. \quad (4.5)$$

The general obstacle in this reduction is the very rapidly increasing number of inequalities. To keep it down we should restrict (4.4) by eliminating redundant inequalities. Denote by $\mathcal{P}_p$ the polyhedron in $\mathbb{R}^{q-p}$ defined by the inequalities obtained as the result of the elimination of $z_0, \ldots, z_{p-1}$. The most economical description of $\mathcal{P}_p$ is the list of its facets. To give this list may be difficult in general but may be possible in our case.

Before we start this reduction in our case, we must add the nonnegativity constraint $x_N \geqslant 0$ to $Fx_N - Gy \geqslant 0$. Our system is than $A^{(0)}z \geqslant 0$, where

$$A^{(0)} = \begin{bmatrix} I & 0 \\ F & -G \end{bmatrix}, \qquad z = \begin{bmatrix} x_N \\ y \end{bmatrix} \quad (4.6)$$

and I is the $(\gamma-\rho)\times(\gamma-\rho)$ identity matrix. Suppose that after the Fourier-Motzkin elimination of $x_{n_0}, \ldots, x_{n_{p-1}}$ we obtained the system $A^{(p)}z^{(p)} \geqslant 0$, where $z^{(p)}$ stands for $(x_{n_p}, \ldots, x_{n_{\gamma-\rho-1}}, y_0, \ldots, y_{\rho-1})^T$. Now by its construction each inequality $A_{i\bullet}^{(p)}z^{(p)} \geqslant 0$ is a positive integer linear combination of some of the original inequalities (whose number t cannot exceed 2^p). Denoting the rows of $A^{(0)}$ corresponding to these inequalities by $u_0, \ldots, u_{t-1}$ and the coefficients by $\alpha_0, \ldots, \alpha_{t-1}$, we can write

$$A_{i\bullet}^{(p)}z^{(p)} = \sum_{i=0}^{t-1} \alpha_i A_{u_i\bullet}^{(0)} z. \quad (4.7)$$

Recall that $\mathcal{P}_p = \{v \in \mathbb{R}^{\gamma-p} : A^{(p)}v \geqslant 0\}$ is the polyhedron obtained by the pth iteration. Setting $c = \gamma - \rho - p$, $U = \{u_0, \ldots, u_{t-1}\}$ and $V = \{s : 0 \leqslant s < c, n_{p+s} \notin U\}$, we have:

LEMMA 4.5. *Let (4.7) hold, and let E' be the submatrix obtained from E by deleting the columns indexed by $u_0, \ldots, u_{t-1}$ and $n_p, \ldots, n_{\gamma-\rho-1}$. Then the dimension of the set $F = \{v \in \mathcal{P}_p : A_{i\bullet}^p v = 0\}$ at most $|V| + \operatorname{rank} E'$.*

Proof. Let $z = (x_{n_p}, \ldots, x_{n_{\gamma-\rho-1}}, y_0, \ldots, y_{\rho-1})^T \in \mathcal{P}_p$ (i.e. $A^{(p)}z \geqslant 0$). By its definition there exist reals $x_{n_0}, \ldots, x_{n_{p-1}}$ such that

$$v = (x_{n_0}, \ldots, x_{n_{\gamma-\rho-1}}, y_0, \ldots, y_{\rho-1})^T$$

satisfies $A^{(0)}v \geqslant 0$. In view of Proposition 4.2 there exist (unique) $x_{b_0},\dots,x_{b_{\rho-1}}$ such that the vector $(x_0,\dots,x_{\gamma-\rho-1})^T$, which is denoted by z^*, satisfies $y=(y_0,\dots,y_{\rho-1})^T=Ez^*$. Moreover by (4.6) and (4.1) the substitution of $y=Ex$ into $A_{j\bullet}^{(0)}v$ yields

$$A_{j\bullet}^{(0)}\begin{bmatrix} x_N \\ Ex \end{bmatrix}=x_j$$

$(j=0,\dots,\gamma-1)$; hence from (4.7) we obtain

$$A_{i\bullet}^{(p)}z=\sum_{i=0}^{t-1}\alpha_i x_{u_i}; \tag{4.8}$$

thus $z\in F$ (i.e. $A_{i\bullet}^{(p)}z=0$) iff $x_{u_0}=\cdots=x_{u_{t-1}}=0$. (Intuitively this shows that for t large there are not enough "free" coordinates to allow the hyperplane to be a facet.) Assume now we have vectors $v_0,\dots,v_{f-1}\in F$ satisfying

$$\sum_{s\in \mathrm{f}}\lambda_s v_s=0 \tag{4.9}$$

for some real $\lambda_0,\dots,\lambda_{f-1}$. For each $s\in\mathrm{f}$ write $w_s=(v_{s0},\dots,v_{s,c-1})^T$ and $y_s=(v_{sc},\dots,v_{s,\gamma-p-1})^T$. By what has been shown above, there exists a nonnegative vector $v_s^*=(x_0,\dots,x_{\gamma-1})^T$ such that $w_s=(x_{n_p},\dots,x_{n_{\gamma-\rho-1}})^T$, $x_{u_0}=\cdots=x_{u_{t-1}}=0$ and $y_s=Ev_s$. Let W be the $c\times f$ matrix with columns $w_0,\dots,w_{f-1}$ (where $c=\gamma-\rho-p$). The first c equations of (4.9) can be written as $W\lambda=0$ where $\lambda=(\lambda_0,\dots,\lambda_{f-1})^T$. If $j\in \mathrm{c}\setminus V$, then $u_{p+j}=n_g$ for some g and, by definition, the jth row of W is the zero vector. Denote by W' the $|V|\times f$ matrix obtained from W by deleting all these zero rows. The first c equations of (4.9) are then replaced by $W'\lambda=0$. Next let X be the $\gamma\times f$ matrix whose columns are $v_0^*,\dots,v_{f-1}^*$. The last ρ equations of (4.9) can be written as $EX\lambda=0$. The rows $u_0,\dots,u_{t-1}$ of X are zero, and $X_{n_{p+j}\bullet}\lambda=0$ because it is one of the equations $W\lambda=0 (s\in\mathrm{c})$. Deleting all these superfluous equations, we get a matrix X', and (4.9) can be written as

$$W'\lambda=0,\qquad E'X'\lambda=0. \tag{4.10}$$

The rank of the matrix of (4.10) cannot exceed rank $W'+\operatorname{rank}E'X'\leqslant|V|+\operatorname{rank}E'$. Hence f is at most $|V|+\operatorname{rank}E'$. ∎

The following proposition shows that for a facet of $\mathcal{P}_p$ the set U contains at most one element of $\{n_p,\dots,n_{\gamma-\rho-1}\}$ (and hence $t=|U|\leqslant p+1$), and the rank of E' is k or $k-1$.

PROPOSITION 4.6. *Let* (4.7) *hold; let* E' *be defined as in Lemma* 4. *Let* $g=|\{U\cap n_p,\dots,n_{\gamma-\rho-1}\}|$, *and let* $\mathcal{P}_p$ *be fully dimensional. If the hyperplane* $\{v\in\mathbb{R}^{\gamma-\rho}: A_i^{(p)}v=0\}$ *is a facet of* $\mathcal{P}_p$, *then* $g\in\{0,1\}$ *and* $\operatorname{rank}E'=\rho-g-1$.

Proof. By Lemma 4.5 we have $|V|+\operatorname{rank}E'=\gamma-p-1$. Clearly $|V|=\gamma-\rho-p-g$. Since E' has ρ rows, the trivial rank inequality gives $\rho\geqslant\operatorname{rank}E'=\gamma-p-1-|V|=\rho+g-1$, proving $g\leqslant 1$. ∎

4.7. In the Fourier-Motzkin elimination the integrality has so far been ignored. Next we briefly mention the discrete version of the method (the method is presented in [22], but is reported to go back to Hilbert and Pressburger). The method is used here to show that, at least in principle, the cone $\operatorname{conv}Y$ can be divided into polyhedral subcones each of which is equipped with a system of linear congruences (not necessarily with the same modulus) such that within each subcone the R-C patterns agree with the integer nonnegative solutions of the corresponding system of congruences.

We start from (4.3). Abbreviate b_{uv}, f_{uv} and g_{uv} by b, f and g. We consider the case $f<g$ only (because the case $f=g$ is trivial and the case $f>g$ is obtained by reversing all inequalities). For fixed $z^{(1)}$ set $F=-A^1_{u*}z^{(1)}$ and $G=A'_{v*}z^{(1)}$. The inequality (4.3) is satisfied by an integer z_0 iff the interval $\lfloor fF,gG\rfloor$ contains an integer multiple of b. Set $s=bg^{-1}$. We distinguish the following $s+1$ separate cases.

(i) Let $b\leqslant gG-fF$. Then clearly $[fF,gG]$ contains an integer multiple of b and we can proceed to (4.4) without any restriction.

(ii) Let $b>gG-fF$, and let $k\in s$ satisfy

$$gk\leqslant gG-fF,\qquad G\equiv k\pmod{s}. \tag{4.11}$$

Then the interval $[fF,gG]$ contains an integer multiple of b [namely $g(G-k)$, which by (4.11) is a multiple of $gs=b$]. Noting that (ii) lists all possibilities for k (mod s), it can be shown easily that (4.11) is the discrete Fourier-Motzkin reduction. [Case (i) is not strictly needed but is included here for the case without congruences.] Thus (4.11) leads to branching or eventually subdivision into polyhedral subcones of $\operatorname{conv}Y$. Note that no branching occurs if $s = 1$. (Congruences with pairwise relatively prime moduli may be combined into a single congruence using the Chinese remainder theorem.)

5. BALANCED ARRAYS

5.1. In this section the separation is discussed for pairs R-C satisfying $R \subseteq C$ and certain interval properties. The motivation comes from balanced two-symbol arrays used in statistics. For $c \in C$ and $r \in R$ set

$$\alpha_{cr} = \sum_{r' \in R \ c \supseteq r' \supseteq r} (-1)^{|r' \setminus r|}. \tag{5.1}$$

By definition $\alpha_{rr} = 1$ for all $r \in R$ and $\alpha_{cr} = 0$ whenever $c \not\supseteq r$. We need the following fact ($\delta_{r'r}$ is the Kronecker δ: $\delta_{r'r} = 1$ if $r' = r$ and $\delta_{rr'} = 0$ otherwise).

Lemma 5.2. *Let $r', r \in R$. Suppose that $r' \supset r$ implies $\alpha_{r'r} = 0$. Then*

$$\sum_{i \in R \ r' \supseteq i \supseteq r} (-1)^{|r' \setminus i|} = \delta_{r'r}.$$

Proof. In view of $|r' \setminus r| = |r' \setminus i| + |i \setminus r|$ for $r' \supseteq i \supseteq r$ and (5.1), the above sum equals $(-1)^{|r' \setminus r|} \alpha_{r'r}$. We know $\alpha_{r'r} = 0$ whenever $r' \not\supseteq r$. The same holds by assumption if $r' \supset r$; hence the sum vanishes unless $r' = r$, in which case it equals 1. ∎

Next the separation is made explicit for special pairs R-C.

Proposition 5.3. *Let $R \subseteq C$, let $N = C \setminus R$, and let $\alpha_{r'r} = 0$ whenever $r' \supset r$. Then $y = E_{RC}x$ is equivalent to the system*

$$x_r = \sum_{r' \in R : r' \supseteq r} (-1)^{|r' \setminus r|} y_{r'} - \sum_{c \in N} \alpha_{cr} x_c \qquad (\forall r \in R). \tag{5.2}$$

Proof. By Proposition 4.1 we know that from $y = Ex$ we can express x_r uniquely as linear combinations of x_c ($c \in N$) and y_r ($r \in R$). To prove the proposition it suffices to plug (5.2) into Ex and verify directly that one obtains y. Thus for $u \in R$ we have

$$\begin{aligned} t_u &= \sum_{p \in C} e_{up} x_p = \sum_{i \in R} e_{ui} x_i + \sum_{j \in N} e_{uj} x_j \\ &= \sum_{i \in R} e_{ui} \Bigl(\sum_{s \in R : s \supseteq i} (-1)^{|s \setminus i|} y_s - \sum_{j \in N} \alpha_{ji} x_j \Bigr) + \sum_{j \in N} e_{uj} x_j . \end{aligned}$$

According to the definition, e_{ui} is 1 if $u \subseteq i$ and 0 otherwise. This and a change of order of summation give

$$t_u = \sum_s y_s \sum_{i: s \supseteq i \supseteq u} (-1)^{|s \setminus i|} + \sum_{j \in N} x_j \Big(e_{uj} - \sum_{i: i \supseteq u} \alpha_{ji} \Big).$$

Now by Lemma 5.2 and (5.1) this simplifies to

$$t_u = \sum_s \delta_{su} y_s + \sum_{j \in N} x_j \Big(e_{uj} - \sum_{i. i \supseteq u} \sum_{s: j \supseteq s \supseteq i} (-1)^{|s \setminus i|} \Big)$$

$$= y_u + \sum_{j \in N} x_j \Big(e_{uj} - \sum_{s: j \supseteq s} \sum_{i: s \supseteq i \supseteq u} (-1)^{|s \setminus i|} \Big)$$

$$= y_u + \sum_{j \in N} x_j \Big(e_{uj} - \sum_{s: j \supseteq s} \delta_{us} \Big)$$

$$= y_u + \sum_{j \in N} x_j (e_{uj} - e_{uj}) = y_u. \qquad \blacksquare$$

Borrowing from the ordered-set terminology, we say that R is *convex* if $r' \subseteq q \subseteq r''$ implies $q \in R$ for all $r', r'' \in R$ and $q \in P$.

The assumption of Proposition 5.3 holds for R a convex subset of C:

COROLLARY 5.4. *Let $R \subseteq C$. If R is convex, then $y = E_{RC} x$ is equivalent to* (5.2).

Proof. Let $p \supset q$. Setting $g = |p \setminus q|$, we have

$$\alpha_{pq} = \sum_{s: p \supseteq u \supseteq q} (-1)^{|u \setminus q|} = \sum_{h=0}^{g} (-1)^h \binom{g}{h} = (1-1)^g = 0,$$

and the statement follows from Proposition 5.3. $\blacksquare$

EXAMPLE 5.5. Consider the very special case of P-intersection patterns (i.e. no restrictions and all intersections are monitored). Obviously P is convex, $N = \varnothing$, and the formulae (5.2) yield

$$x_i = \sum_{s\ s \supseteq i} (-1)^{|s \setminus i|} y_s.$$

This inversion formula is of the inclusion-exclusion type, i.e. a Möbius inversion.

5.6. Now we consider another interesting class of pairs R-C. We say that R-C has *uniform intervals* if $R \subseteq C$ and for each pair $r \subset c$ ($r \in R$, $c \in C$) there is an integer ε_{rc} such that for all $r \subseteq q \subseteq c$ we have

$$q \in R \quad \Leftrightarrow \quad |q| \leqslant \varepsilon_{rc}$$

(i.e., in the interval $[r,c]$ membership in R depends solely on the cardinalities). For R-C with uniform intervals we have

$$\alpha_{cr} = \sum_{l=0}^{|c\setminus r|} (-1)^l \binom{\varepsilon_{rc} - |r|}{l} \tag{5.3}$$

whenever $c \supset r$.

5.7. We study in detail the special case of P_t-patterns which are essentially the two-symbol balanced arrays used in statistics. Let m and t be positive integers, $m \geqslant t$, and let $\mu_0, \ldots, \mu_t$ be nonnegative integers. A binary $m \times n$ matrix X such that each $t \times n$ submatrix of X contains every binary column vector $(z_1, \ldots, z_t)^T$ with $z_1 + \ldots + z_t = i$ precisely μ_i times $(i = 0, \ldots, t)$ is called an $m \times n$ *balanced two-symbol array of strength* t *and indices* $\langle \mu_0, \ldots, \mu_t \rangle$ (BA for short). Thus a BA is characterized by the fact that in any set of t rows the frequency of each column depends exclusively on the number of 1's it contains. The BA were considered by Srivastava [21] among others. A list of the rather extensive literature can be found in [16]. If the strength t equals 1, then the BA is the exact cover induced in game theory by Shapley [20] as balanced sets and studied by Graver and Jurkat, who also considered the case $t > 1$ under the name of general t-design [11].

If we partition the columns of X according to the binary column vectors in the first t rows, we arrive at once at

$$n = \sum_{l=0}^{t} \binom{t}{l} \mu_l. \tag{5.4}$$

Hence $\mu_0, \mu_1, \ldots, \mu_t$, and n are not independent. To avoid redundancy we omit μ_0 from the indices. Moreover the zero columns of X may be deleted, with the result that n and μ_0 are reduced by the same amount. Thus for simplicity we assume that X has no zero columns. We exhibit now the close relationship between the BA and certain P_t-intersection patterns. We say that a P_t-pattern y is *homogeneous* with indices $\langle\lambda_1,\dots,\lambda_t\rangle$ if $y_r=\lambda_j$ whenever $|r|=j$ ($r\in R$, $j\in\{1,\dots,t\}$). We have:

PROPOSITION 5.8. *Let X be an $m\times n$ binary matrix without zero columns. Then X is a balanced two-symbol array of strength t and index sequence $\langle\mu_1,\dots,\mu_t\rangle$ if and only if the corresponding P_t-intersection pattern is a homogeneous one with $\langle\lambda_1,\dots,\lambda_t\rangle$ related to $\langle\mu_1,\dots,\mu_t\rangle$ by the formulae*

$$\lambda_h=\sum_{l=0}^{t-h}\binom{t-h}{l}\mu_{h+l}\qquad (h=1,\dots,t)\tag{5.5}$$

$$\mu_h=\sum_{l=0}^{t-h}(-1)^l\binom{t-h}{l}\lambda_{h+l}\qquad (h=1,\dots,t).\tag{5.6}$$

Proof. Necessity: For $1\leqslant h\leqslant t$, $0\leqslant l\leqslant t-h$ and a binary vector $z=(z_1,\dots,z_t)$ with $z_1+\cdots+z_t=h+l$, there are exactly μ_{h+l} columns of X having z in the first t rows. Thus for $h\leqslant t$ the number of columns having the h-vector $(1,\dots,1)^T$ in the first h rows is

$$\lambda_h=\sum_{l=0}^{t-h}\binom{t-h}{l}\mu_{h+l}.$$

Since this number does not change if we replace the first h rows by any set of h rows, the corresponding P_t-pattern is indeed a homogeneous one with parameters $\lambda_1,\dots,\lambda_t$.

Sufficiency: Let X be an $m\times n$ binary matrix such that the corresponding P_t-pattern is homogeneous with parameters $\lambda_1,\dots,\lambda_t$. For $q\subseteq \mathbf{m}$ set $q^{\to}=(q_0,\dots,q_{m-1})^T$, where $q_i=1$ if $i\in q$ and $q_i=0$ otherwise ($i\in\mathbf{m}$). Further let x_q denote the frequency of $q^{\to}$ as a column vector of X. For the ease of notation consider the first t rows of X and the binary column t-vector $z=(1,\dots,1,0,\dots,0)^T$ with h entries 1. Set $r=\{0,\dots,h-1\}$ and $s=\{0,1,\dots,h-1,t,t+1,\dots,m-1\}$. Then the frequency of z in the first t rows of X is

$$\sum_{p:s\supseteq p\supseteq r}x_p.$$

For $i=1,\dots,t-h$ set $U_i=\{p\in P:\{0,\dots,t-1\}\supset p\supset r,\ |p|=h+i\}$. Applying the principle of inclusion and exclusion,

$$\sum_{s\supseteq p\supseteq r} x_p = \sum_{p\supseteq r} x_p - \sum_{u\in U_1}\sum_{p\supseteq u} x_p + \cdots$$

$$+(-1)^{t-h}\sum_{u\in U_{t-h}}\sum_{p\supseteq u} x_p \tag{5.7}$$

(because for $p\supseteq r$ with $|p\cap\{h,\ldots,t-1\}|=j$, the symbol x_p appears once in the first sum, $\binom{j}{1}$ times in the second, etc., and hence altogether with the coefficient 0 if $j>0$ and 1 if $j=0$). By definition and homogeneity for $i=1,\ldots,t$ and $q\subseteq$ m we have $\Sigma_{p\supseteq q}x_p=\lambda_{|q|}$; hence using

$$|U_i| = \binom{t-h}{i}$$

we get from (5.7)

$$\sum_{s\supseteq p\supseteq r} x_g = \sum_{i=0}^{t-h}(-1)^i\binom{t-h}{i}\lambda_{h+i}. \tag{5.8}$$

Obviously the same result is obtained choosing t arbitrary rows of X and h arbitrary rows among these t rows. Thus the right side of (5.8) equals μ_h ($h=1,\ldots,t$). Observing that μ_0 is determined by (5.4), this yields that X is a BA with the parameters given by (5.5). ■

5.9. The last proposition allows us to consider BAs as homogeneous P_t-patterns. Clearly in this case it has uniform intervals ($\varepsilon_{rc}=t$ for all pairs $c\supset r$). Using Proposition 5.4, (5.2), $y_r=\lambda_{|r|}$, $R=P_t$, and setting $j=|r|$, we get from (5.2)

$$x_r = \sum_{h=0}^{t-j}(-1)^h\binom{m-j}{h}\lambda_{j+h} - \sum_{c\ |c|>t,\, c\supset r} x_c\sum_{l=0}^{t-j}(-1)^l\binom{|c|-j}{l}.$$

Apply (5.5) to eliminate λ_{j+h}. Setting $h+l=q$, the first sum may be transformed into

$$\sum_{q=0}^{t-j}\mu_{j+q}\sum_{h=0}^{q}(-1)^h\binom{m-j}{h}\binom{t-j-h}{q-h}.$$

Using a well-known identity (cf. [17, Sec. 1.3, (5), p. 8]) the inside sum equals

$$(-1)^q\binom{m+q-t-1}{q},$$

hence we obtain

$$\begin{aligned} x_r = {} & \sum_{q=0}^{t-j}(-1)^q\binom{m+q-t-1}{q}\mu_{j+q} \\ & - \sum_{c\,:\,|c|>t,\, c\supset r} x_c \sum_{l=0}^{t-j}(-1)^l\binom{c-j}{l}. \end{aligned} \tag{5.9}$$

Recall that for $c\subseteq \mathbf{m}$ we set $\chi_{\mathbf{m}}(c)=(c_0,\ldots,c_{m-1})^T$, where $c_i=1 \Leftrightarrow i\in c$ and $c_i=0$ otherwise. The transition to inequalities yields:

PROPOSITION 5.10. *Let m and t be positive integers, $m\geqslant t$, and let $\mu_1,\ldots\mu_t$ be nonnegative integers. Then there exists an m-row two-symbol balanced array of strength t and indices $\langle\mu_1,\ldots,\mu_t\rangle$ if and only if the system of inequalities*

$$\sum_{c\,:\,|c|>t,\, c\supset r} x_c \sum_{l=0}^{t-|r|}(-1)^l\binom{|c|-|r|}{l} \leqslant \sum_{l=0}^{t-|r|}(-1)^l\binom{m+l-t-1}{l}\mu_{|r|+l} \tag{5.10}$$

($\forall r\subseteq \mathbf{m}, |r|\leqslant t$) is solvable in nonnegative integers x_c ($c\subseteq \mathbf{m}, |\mathbf{c}|>\mathbf{t}$). Moreover, if we have such solution of (5.9) and x_r ($r\in R$) are determined from (5.8), then each m-row matrix having exactly x_c columns $\chi_{\mathbf{m}}(c)=(c_0,\ldots,c_{m-1})^T$ is a balanced array of our type, and all such balanced arrays are obtainable in this way.

5.11. For given $\mu_1,\ldots,\mu_t$ the solution of (5.10) in nonnegative x_c is a typical integer linear-programming problem. The number of variables x_c is

$$\binom{m}{t+1}+\cdots+\binom{m}{m},$$

which grows fast as t gets smaller. It is natural to start with t close to m. The case $t=m$ being trivial, we consider $t=m-1$. In this case there is but one variable $x = x_m$ in (5.10), which easily simplifies to

$$(-1)^{m-q-1}x \leqslant \sum_{l=0}^{m-q-1} (-1)^l \mu_{q+l} \qquad (q=1,\dots,m-1).$$

6. PATTERNS AND FORMS

In this section we discuss the R-patterns only (i.e. patterns without any restriction on the support). To a given family $\mathcal{C} = \{A_0,\dots,A_{m-1}\}$ associate the zero-one matrix $X_{\mathcal{C}} = (x_{ij})$ (with $x_{ij} = 1$ if $a_j \in A_i$ and $x_{ij} = 0$ otherwise). In terms of x_{ij} we obtain: the sequence $y \in \mathbb{N}^p$ is an R-pattern if and only if

$$y_r = \sum_{j\in \mathbf{n}} \prod_{p\in r} x_{pj} \qquad (r\in R) \tag{6.1}$$

for some $n \in P$ and $m\times n$ zero-one matrix $X = (x_{ij})$. In this section it will be more convenient to switch from the set R (of subsets of $\mathbf{m}$) to a set H of h-tuples of elements of $\mathbf{m}$ (i.e. $H \subseteq \bigcup_{p=1}^{\rho} \mathbf{m}^p$). There is no harm in this change, because coordinates can be repeated (this is based on $x^2 = x$ for $x\in \mathbf{2} = \{0,1\}$) and various reorderings of coordinates lead to no more than a certain redundancy. As an example we have the transition from the (pairwise) intersection patterns (where R is the set P_2 of all singletons and pairs from $\mathbf{m}$) to the intersection matrices (where H is the set $\mathbf{m}^2$ of all pairs from $\mathbf{m}$). For the simplicity of notation the coordinate of y corresponding to $(i_1,\dots,i_h)\in H$ is denoted $y_{i_1\dots i_h}$. With this notation (6.1) becomes

$$y_{i_1\dots i_h} = \sum_{j\in\mathbf{n}} x_{i_1 j}\cdots x_{i_h j}, \qquad (i_1,\dots,i_h)\in H. \tag{6.2}$$

Note that the homogeneous polynomials in (6.2) depend on the way R was converted into H; e.g., if H is ternary the set $\{1,2\}\in R$ may yield either $x_1^2x_2$ or $x_1x_2^2$. Although all these polynomials agree on $\{0,1\}$, they are no longer identical if the variables are not binary.

Now we relax the stipulation "$X = (x_{ij})$ binary" by allowing x_{ij} to range over a given subset S of $\mathbb{C}$ (the sets S of interest being $\mathbf{2}$, $\{-1,1\}$, $\mathbb{N}$, $\mathbb{R}$, $\mathbb{Q}_+$, $\mathbb{Q}$, $\mathbb{R}_+$ and $\mathbb{R}$). Thus let T_S be the set of all, sequences (indexed by H) $\langle y_{i_1\dots i_h} : (i_1,\dots,i_h)\in H\rangle$ satisfying (6.2) for some $m\times n$ matrix (x_{ij}) over S (in particular $Y = T_2$). This relaxation, of a type quite common in integer programming, is introduced with the hope that some T_s may provide useful information about Y. Clearly $T_{S'} \subseteq T_{S''}$ whenever $S' \subseteq S''$. Denote by B_S the subset of T_S corresponding to $m \times 1$ matrices X over S, and set $r = |H|$. We have:

PROPOSITION 6.1. *For each $S \subseteq R$ the set T_S with the vector addition is an monoid generated by B_S. The set $T_{\mathbb{R}_+}$ is a pointed convex cone (with apex 0) in $\mathbb{R}^r$, while $T_{\mathbb{R}}$ is a subspace of $\mathbb{R}^r$.*

Proof. If y' and y'' correspond to $X' \in S^{m \times n'}$ and $X'' \in S^{m \times n''}$, then $y' + y''$ corresponds to $[X', X''] \in S^{m \times (n' + n'')}$. The other statements are also easily proved. ■

The dual cone to T_S is denoted by D_S. and its minimal basis by $\bar{D}_S$. Thus $d \in D_S$ belongs to $\bar{D}_S$ if the hyperplane $\{x \in \mathbb{R}^r : dx = 0\}$ is a facet of T_S (i.e. shares with T_S a set of $\dim T_S - 1$ linearly independent vectors). It is well known that the transition to dual cones reverses inclusions; hence $D_{S'} \supseteq D_{S''}$ whenever $S' \subseteq S''$. We have

LEMMA 6.2. *If $S' \subseteq S''$ while $\dim T_{S'} = \dim T_{S''}$, then $\bar{D}_{S'} \cap D_{S''} \subseteq \bar{D}_{S''}$.*

Proof. Let $\delta = \dim T_{S'}$ and let $d \in \bar{D}_{S'} \cap D_{S''}$. By definition there is a set of $\delta - 1$ linearly independent vectors shared by $T_{S'}$ and $\{x \in \mathbb{R}^r : dx = 0\}$. Clearly these vectors belong to $T_{S''}$ and thus cause $d \in D_{S''}$ to belong to $\bar{D}_S$. ■

For $H = \mathrm{m}^2$ the set $D_{\mathbb{R}_+}$ (i.e. the minimal basis for the dual cone to $\{XX^T : X \text{ an } m\text{-row nonnegative matrix}\}$) has been studied; however, from its known elements listed in [13, Chapter 16.2], only the vectors $d = (d_{11}, \ldots, d_{mm})$, whose only nonzero coordinates are $d_{ij} = d_{ji} = 1$ $(1 \leqslant i < j \leqslant m)$ are shared by $\bar{D}_R$ and $D_{\{0,1\}}$.

Membership in D_s amounts to the nonnegativity of a certain polynomial over S^m. For this purpose, to $d = \langle d_{i_1 \ldots i_h} : \langle i_1, \ldots, i_h \rangle \in H \rangle$ associate the polynomial

$$f_d(z_1, \ldots, z_m) = \sum_{\langle i_1, \ldots, i_h \rangle \in H} d_{i_1 \ldots i_h} z_{i_1} \cdots z_{i_h} \tag{6.3}$$

in real variables $z_1, \ldots, z_m$. We say that d is *S-copositive* if f_d assumes nonnegative values on S^m. For $\rho = \mathrm{m}^2$ the polynomial f_d is nothing else than the quadratic form $\sum_{i_1, i_2 \in \mathrm{m}} d_{i_1 i_2} z_{i_1} z_{i_2}$ corresponding to the matrix d. In this case $\mathbb{R}_+$-copositivity is simply the standard copositivity (i.e. nonnegativity of

f_d on the nonnegative octant), and $\mathbb{R}$-copositivity means that d and f_d are positive semidefinite. We have:

PROPOSITION 6.3. *Let $d \in \mathbb{R}^r$. Then $d \in D_S$ if and only if d is S-copositive.*

Proof. Necessity: Let $z_1, \dots, z_m \in S$. Then y defined by $y_{i_1 \dots i_h} = z_{i_1} \cdots z_{i_h}$ for all $\langle i_1, \dots, i_h \rangle \in H$ belongs to T_S; hence

$$dy = \sum_{\langle i_1, \dots, i_h \rangle \in H} d_{i_1 \dots i_h} z_{i_1} \cdots z_{i_h} \geq 0$$

by the definition of D_S.

Sufficiency: Let $y \in T_S$. By (6.2) and the S-copositivity of d.

$$dy = \sum_{\langle i_1, \dots, i_h \rangle \in H} d_{i_1 \dots i_h} \sum_{l \in \mathbf{n}} x_{i_1 l} \cdots x_{i_h l}$$

$$= \sum_{l \in \mathbf{n}} f_d(x_{1l}, \dots, x_{nl}) \geq 0,$$

proving $d \in D_S$. ∎

To follow the analogy with Y and I, set $Z = \bigcup_{p=1}^{\infty} p^{-1} T_2$. To prove that $Z \subseteq T_{\mathbb{Q}_+}$ we need the additional assumption $H \subseteq \mathbf{m}^h$ (i.e. H is an h-ary relation).

PROPOSITION 6.4. *If $H \subseteq \mathbf{m}^h$, then $Z \subseteq T_{\mathbb{Q}+}$.*

Proof. Let $p \in \mathbb{P}$, and let (x_{ij}) be an $m \times n$ binary matrix. Let $t = p^{h-1} n$, let (y_{ij}) be an $m \times t$ binary matrix obtained from (x_{ij}) by duplicating each of its columns p^{h-1} times, and finally let $(w_{ij}) = p^{-1}(y_{ij})$. Clearly $(w_{ij}) \in \mathbb{Q}^{m \times t}$, and for each $\langle i_1, \dots, i_h \rangle \in H$

$$p^{-1} \sum_{l \in \mathbf{n}} x_{i_1 l} \cdots x_{i_h l} = p^{-h} \sum_{l \in \mathbf{n}} p^{h-1} x_{i_1 l} \cdots x_{i_h l}$$

$$= p^{-h} \sum_{q \in \mathbf{t}} y_{i_1 q} \cdots y_{i_h q} = \sum_{q \in \mathbf{t}} w_{i_1 q} \cdots w_{i_h q},$$

proving $p^{-1}(x_{ij}) \in T_{\mathbb{Q}_+}$ and $Z \subseteq T_{\mathbb{Q}_+}$. ∎

The integer sequences from Z can be described in a slightly different way. Let Z' be the set of all $\langle y_{i_1\ldots i_h} \cdot \langle i_1,\ldots,i_h\rangle \in H\rangle$ such that

$$y_{i_1\ldots i_h} = \sum_{l\in \mathbf{n}} c_l x_{i_1 l}\cdots x_{i_h l} \qquad (\langle i_1,\ldots,i_h\rangle \in H) \tag{6.4}$$

for an $n\in\mathbb{P}$, $c_1,\ldots,c_n\in\mathbb{R}_+$ and a binary $m\times n$ matrix X (in other words, it corresponds to matrices Xc where $X\in 2^{m\times n}$ and $c\in\mathbb{R}_+^n$). This use of column weights was suggested by [15], where it was used in the special case $H=\mathbf{m}^2$; $n=m$ and $y_{ij}=\lambda+(k-\lambda)\delta_{ij}$ $(i,j\in\mathbf{m})$.

For h-ary relations this creates no rational sequences outside Z.

Proposition 6.5. *If $H\subseteq \mathbf{m}^h$, then $Z=Z'\cap\mathbb{Q}^r$.*

Proof. Let $y\in Z'\cap\mathbb{Q}^r$ satisfy (6.4) for $c_1,\ldots,c_n\in\mathbb{R}_+$ and $(x_{ij})\in 2^{m\times n}$. Viewing (6.4) as a system of linear algebraic equations for $c_1,\ldots,c_n$ with integer coefficients we may assume that c_i rational: $c_i=p_iq_i^{-1}$ $(p_i\in\mathbb{N}, q_i\in\mathbb{P}, i=1,\ldots,n)$. Set

$$q = q_1\cdots q_n; \qquad t_i = qp_iq_i^{-1} \ (i=1,\ldots,n) \quad \text{and} \quad t = t_1+\cdots+t_n.$$

Let (y_{ij}) be the zero-one $m\times t$ matrix obtained from (x_{ij}) by repeating the ith column z_i times $(i=1,\ldots,n)$. Clearly

$$y_{i_1\ldots i_h} = q^{-1}\sum_{l\in\mathbf{n}} z_l x_{i_1 l}\cdots x_{i_h l} = q^{-1}\sum_{u\in \mathbf{t}} y_{i_1 u}\cdots y_{i_h u},$$

proving $y\in Z$. The other inclusion being obvious, this concludes the proof. ∎

Example 6.6. Now we turn to the special case $\rho=\mathbf{m}^h$. Suppose $z\in T_S$. Substituting the corresponding $(x_{ij})\in S^{m\times n}$ [satisfying (6.2)] into (6.3) and using the symmetry, we obtain

$$f_y(z_1,\ldots,z_n) = \sum_{j\in\mathbf{n}} (x_{1j}z_1+\cdots+x_{mj}z_m)^h. \tag{6.5}$$

Conversely, if f_y decomposes into a sum of h-powers of linear homogeneous expressions (6.5), then y and (x_{ij}) satisfy (6.2); thus if all $x_{ij}\in S$, then $y\in T_S$. For example, if all the linear expressions in (6.5) turn out to be just sums of variables, then $y\in T_2$. If all the x_{ij} belong to $\mathbb{R}_+$, then, of course, f_y is

$\mathbb{R}_+$-copositive (it is called completely positive if $H = m^2$, i.e. if f_y is a quadratic form).

For an even h, clearly f_y is $\mathbb{R}$-copositive (i.e. a positive semidefinite form if $h = 2$).

In this context the set $T_{\mathbb{R}}$ was first considered by Schoenberg [19]. Connor used the fact that f_y must be a positive semidefinite form to prove the nonexistence of certain (v, b, r, k, λ) block designs [1]. Also for $H = m^2$ the sets T_2, Z, $T_{\mathbb{Q}_+}$, $T_{\mathbb{R}_+}$, $T_{\mathbb{Q}}$ and $T_{\mathbb{R}}$ are discussed in [7], and the inclusions between them are shown to be proper. By a classical result, each positive semidefinite form f_y decomposes into a sum of squares of linear expressions; hence by the result quoted above we have $y \in T_{\mathbb{R}}$. It follows from Proposition 6.3 that for $H = m^2$ we have $D_{\mathbb{R}} = T_{\mathbb{R}}$ (i.e. the set $T_{\mathbb{R}}$ is self-dual).

REMARK 6.7. In block designs sometimes T_2 is restricted by requiring that (x_{ij}) be a square matrix. Thus one considered the set

$$T_2^1 = \{XX^T : X \in 2^{n \times n}\}.$$

Obviously this definition can be made for an arbitrary S and H.

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